

ON THE GENERALIZED QUADRATIC MAPPINGS IN QUASI-BANACH MODULES OVER A C^* -ALGEBRA *

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ABSTRACT. Let $n > 2$ be a positive integer. In this paper, we obtain the general solution of the following functional equation

$$n \sum_{1 \leq i < j \leq n} Q(x_i - x_j) = \sum_{i=1}^n Q\left(\sum_{j=1}^n x_j - nx_i\right)$$

which is derived from the centroid of the n distinct vectors x_1, \dots, x_n in an inner product space. Furthermore, we prove that a mapping f between quasi-Banach modules over a C^* -algebra satisfying approximately the equation can be approximated by a quadratic mapping Q satisfying exactly the equation such that $\|f(x) - Q(x)\|$ is bounded.

1. INTRODUCTION

The stability problem of functional equations originated from a question of S.M. Ulam [25] concerning the stability of group homomorphisms: “When is it true that by slightly changing the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true?” If the answer is affirmative, then we would say the equation of homomorphism $H(x * y) = H(x) \diamond H(y)$ is stable. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation.

First, Ulam’s question for approximately additive mappings was solved by D.H. Hyers [10]. In 1951, D.G. Bourgin [4] was the second author to treat the Ulam stability problem for additive mappings. Th.M. Rassias [18] succeeded in extending the result of Hyers’ theorem by weakening the condition for the Cauchy difference to be unbounded. A number of mathematicians were attracted to this result of Th.M. Rassias and stimulated to investigate the stability problems of functional equations.

Now, a square norm on an inner product space satisfies the important parallelogram equality $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$ for all vectors x, y . If $\triangle ABC$ is a triangle in a finite dimensional Euclidean space and I is the center of the side \overline{BC} , then the following identity $\|\overrightarrow{AB}\|^2 + \|\overrightarrow{AC}\|^2 = 2(\|\overrightarrow{AI}\|^2 + \|\overrightarrow{CI}\|^2)$ holds for all vectors A, B and C . The following functional equation, which was motivated by

1991 *Mathematics Subject Classification.* 39B82, 46L05, 30D05.

Key words and phrases. Ulam stability problem, A-quadratic mapping, Unitary group, quasi-Banach modules, p-Banach modules.

This Article was submitted in The Journal of Mathematical Analysis and Applications

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these equations,

$$(1.1) \quad Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y)$$

is called a *quadratic functional equation*, and every solution of the equation (1.1) is said to be a *quadratic mapping*. A Hyers-Ulam stability problem for the quadratic functional equation (1.1) was first solved by F. Skof [23]. C. Borelli and G.L. Forti [3] generalized the stability result of the quadratic functional equation. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem [1, 7, 9, 11, 19]. Furthermore, C. Park [17] have proved the Hyers-Ulam-Rassias stability problem for functional equations in Banach modules over a C^* -algebra.

Now, if $\triangle XYZ$ is a triangle in a finite dimensional Euclidean space and $G := \frac{X+Y+Z}{3}$ is the center of gravity of the triangle, then a simple direct calculation and the definition of the norm yields the following identity

$$(1.2) \quad \|\overrightarrow{XY}\|^2 + \|\overrightarrow{YZ}\|^2 + \|\overrightarrow{ZX}\|^2 = 3 \left(\|\overrightarrow{XG}\|^2 + \|\overrightarrow{YG}\|^2 + \|\overrightarrow{ZG}\|^2 \right).$$

Employing the above identity (1.2), we introduce the new functional equation,

$$(1.3) \quad \begin{aligned} & 3Q(x-y) + 3Q(y-z) + 3Q(x-z) \\ & = Q(y+z-2x) + Q(x+z-2y) + Q(x+y-2z) \end{aligned}$$

for a mapping $Q : U \rightarrow V$ and for all vectors $x, y, z \in U$, where U and V are linear spaces. More generally, let X_1, X_2, \dots, X_n ($n \geq 3$) be distinct vectors in a finite dimensional Euclidean space E . Putting $G := \frac{\sum_{i=1}^n X_i}{n}$, the centroid of the n distinct vectors, then we get the following identity by a simple direct calculation and the definition of the norm

$$\sum_{1 \leq i < j \leq n} \|\overrightarrow{X_i X_j}\|^2 = n \sum_{i=1}^n \|\overrightarrow{X_i G}\|^2$$

which is equivalent to the equation

$$(1.4) \quad n \sum_{1 \leq i < j \leq n} \|X_i - X_j\|^2 = \sum_{i=1}^n \left\| \sum_{j=1}^n X_j - nX_i \right\|^2$$

for any distinct vectors X_1, X_2, \dots, X_n . Employing the above equality (1.4), we introduce the new functional equation,

$$(1.5) \quad n \sum_{1 \leq i < j \leq n} Q(x_i - x_j) = \sum_{i=1}^n Q \left(\sum_{j=1}^n x_j - nx_i \right)$$

for a mapping $Q : U \rightarrow V$ and for all vectors $x_1, \dots, x_n \in U$.

We recall some basic facts concerning quasi-Banach spaces and some preliminary results.

Definition 1.1. ([2, 20]) Let X be a linear space. A *quasi-norm* $\|\cdot\|$ is a real-valued function on X satisfying the following:

- (1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.

(2) $\|\lambda x\| = |\lambda| \cdot \|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$.

(3) There is a constant K such that $\|x + y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in X$.

The smallest possible K is called the *modulus of concavity* of $\|\cdot\|$. The pair $(X, \|\cdot\|)$ is called a *quasi-normed space* if $\|\cdot\|$ is a quasi-norm on X . A *quasi-Banach space* is a complete quasi-normed space. A quasi-norm $\|\cdot\|$ is called a *p-norm* ($0 < p \leq 1$) if

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$. In this case, a quasi-Banach space is called a *p-Banach space*.

Clearly, p -norms are continuous, and in fact, if $\|\cdot\|$ is a p -norm on X , then the formula $d(x, y) := \|x - y\|^p$ defines an translation invariant metric for X and $\|\cdot\|^p$ is a p -homogeneous F -norm. The Aoki–Rolewicz theorem [2, 20] guarantees that each quasi-norm is equivalent to some p -norm for some $0 < p \leq 1$. Concerning the Ulam stability problem for functional equations, C. Sánchez [21] and J. Tabor [24] have investigated a version of the Hyers-Rassias-Gajda theorem (see [8, 18]) for approximate additive mappings in quasi-Banach spaces.

In this paper, we are going to find the general solution of (1.5) for any fixed positive integer $n \geq 3$ in the class of mappings between real vector spaces. Furthermore, concerning the stability problem of Ulam for the functional equation (1.5) we are going to investigate the generalized Hyers-Ulam-Rassias stability problem for approximate mappings in quasi-Banach modules and p -Banach modules over a C^* -algebra. Thus we generalize the stability results of the quadratic functional equation (1.5) in Banach spaces.

2. SOLUTION OF FE. (1.5)

First of all, we find out the general solution of (1.3) in the class of mappings between real vector spaces.

Lemma 2.1. *Let U and V be real vector spaces. A mapping $Q : U \rightarrow V$ satisfies the functional equation (1.3) if and only if the mapping $Q : U \rightarrow V$ is quadratic.*

Proof. It is easy to see that the equation (1.1) implies the functional equation (1.3). Now let Q satisfy the equation (1.3). Putting $y, z := 0$ in (1.3) yields $Q(2x) = 4Q(x)$ for all $x \in U$. By setting $z := 0$ in (1.3), we see

$$(2.1) \quad Q(x - 2y) + Q(2x - y) + Q(x + y) = 3Q(x - y) + 3Q(x) + 3Q(y)$$

for all $x, y \in U$. In turn, substituting $-y$ for y in (2.1) and then adding the resulting equation to (2.1), one obtains

$$(2.2) \quad \begin{aligned} & Q(2x + y) + Q(2x - y) + Q(x + 2y) + Q(x - 2y) \\ &= 2Q(x - y) + 2Q(x + y) + 6Q(x) + 6Q(y) \end{aligned}$$

for any $x, y \in U$. Letting $z := -y$ in (1.3), we obtain

$$(2.3) \quad Q(x + 3y) + Q(x - 3y) + 4Q(x) = 3Q(x - y) + 3Q(x + y) + 12Q(y)$$

for all $x, y \in U$. Replacing x by $2x$ in (2.3), we get

$$(2.4) \quad \begin{aligned} Q(2x + 3y) + Q(2x - 3y) + 16Q(x) \\ = 3Q(2x - y) + 3Q(2x + y) + 12Q(y) \end{aligned}$$

for any $x, y \in U$. Now we substitute $z := 2y$ in (1.3) to get

$$(2.5) \quad Q(x - 3y) + Q(2x - 3y) + Q(x) = 3Q(x - y) + 3Q(y) + 3Q(x - 2y)$$

for any $x, y \in U$. Replacing y by $-y$ in (2.5) and then adding (2.5) to the resulting expression, we obtain

$$\begin{aligned} Q(x + 3y) + Q(x - 3y) + Q(2x + 3y) + Q(2x - 3y) + 2Q(x) \\ = 3Q(x + y) + 3Q(x - y) + 3Q(x + 2y) + 3Q(x - 2y) + 6Q(y), \end{aligned}$$

which is rearranged in the following way by (2.4)

$$(2.6) \quad \begin{aligned} Q(x + 3y) + Q(x - 3y) + 3Q(2x + y) + 3Q(2x - y) + 6Q(y) \\ = 3Q(x + y) + 3Q(x - y) + 3Q(x + 2y) + 3Q(x - 2y) + 14Q(x) \end{aligned}$$

for any $x, y \in U$. Now subtracting (2.3) from the equation (2.6) and then dividing it by 3, we have

$$(2.7) \quad Q(2x + y) + Q(2x - y) + 6Q(y) = Q(x + 2y) + Q(x - 2y) + 6Q(x)$$

for any $x, y \in U$. Again we add (2.2) to (2.7) and then divide the resulting expression by 2 to obtain

$$(2.8) \quad Q(2x + y) + Q(2x - y) = Q(x + y) + Q(x - y) + 6Q(x),$$

which is equivalent to the original quadratic functional equation

$$Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y)$$

for any $x, y \in U$ [5, Theorem 2.1]. □

Lemma 2.2. *Assume that a mapping $Q : U \rightarrow V$ satisfies the functional equation (1.5). Then Q is even and*

$$(2.9) \quad Q((n - 1)^k x) = (n - 1)^{2k} Q(x)$$

for any vector $x \in U$.

Proof. By setting $x_i := 0$ for all $i = 1, \dots, n$ in the equation (1.5), we see $Q(0) = 0$. Putting $x_1 = x$ and $x_i := 0$ for all $i = 2, \dots, n$ in (1.5), we get $Q(-(n - 1)x) = (n - 1)^2 Q(x)$ for all $x \in U$. Substituting $x_i := x$ for all $i = 1, \dots, n - 1$ and $x_n := 0$ in (1.5), one obtains

$$n(n - 1)Q(x) = (n - 1)Q(-x) + Q((n - 1)x) = n(n - 1)Q(-x),$$

which shows that Q is even, and hence $Q((n - 1)x) = (n - 1)^2 Q(x)$ for all $x \in U$. Therefore we get the desired conclusion by induction on k . □

To find the general solution of (1.5), we need to prove the following lemma above all.

Lemma 2.3. *Let U and V be real vector spaces. For each integer a with $|a| \neq 1$, a mapping $Q : U \rightarrow V$ satisfies the functional equation*

$$(2.10) \quad \begin{aligned} Q(ax + y) + Q(x + ay) + (a - 1)Q(x - y) \\ = (a + 1)Q(x + y) + (a^2 - 1)[Q(x) + Q(y)] \end{aligned}$$

for all $x, y \in U$ if and only if a mapping $Q : U \rightarrow V$ is quadratic.

Proof. Let Q satisfy the equation (2.10). It follows easily that Q is even, $Q(ax) = a^2Q(x)$ and $Q(0) = 0$. For $a = 0$, the equation (2.10) reduces to the equation (1.1). For any negative integer $a < -1$, by considering a as $-a$ and applying the evenness of Q , we need to prove the lemma for the case $a > 1$ without loss of generality. Now we claim that if Q satisfies the equation (2.10), then Q also satisfies (1.1) by induction on positive integers $a > 1$. For $a = 2$, the equation (2.10) reduces to

$$(2.11) \quad Q(2x + y) + Q(x + 2y) + Q(x - y) = 3Q(x + y) + 3Q(x) + 3Q(y),$$

which is exactly the equation (1.3), and hence it is equivalent to (1.1) by Lemma 2.1. Assume that the equation (2.10) implies the equation (1.1) for all a with $a := 2, \dots, a$. We are to show that if Q satisfies the equation (2.10) for $a + 1$, then Q is quadratic in the sequel. Letting $y := x + y$ in (2.10), we obtain

$$(2.12) \quad \begin{aligned} Q((a + 1)x + y) + Q((a + 1)x + ay) + (a - 1)Q(y) \\ = (a + 1)Q(2x + y) + (a^2 - 1)[Q(x) + Q(x + y)] \end{aligned}$$

for all $x, y \in U$. Interchanging x with y in (2.12) and after that adding it to (2.12), we have

$$(2.13) \quad \begin{aligned} Q((a + 1)x + y) + Q(x + (a + 1)y) + Q((a + 1)x + ay) + Q(ax + (a + 1)y) \\ = (2a^2 + 3a + 1)Q(x + y) + (a^2 + 2a + 3)[Q(x) + Q(y)] - (a + 1)Q(x - y) \end{aligned}$$

for all $x, y \in U$. Letting $y := -x + y$ in (2.10), we obtain

$$(2.14) \quad \begin{aligned} Q((a - 1)x + y) + Q((a - 1)x - ay) + (a - 1)Q(2x - y) \\ = (a + 1)Q(y) + (a^2 - 1)[Q(x) + Q(x - y)] \end{aligned}$$

for all $x, y \in U$. Exchanging x and y in (2.14) and after that adding the resulting equation and (2.14), one has by induction

$$(2.15) \quad \begin{aligned} Q((a - 1)x + ay) + Q(ax + (a - 1)y) + Q(x - y) \\ = (2a^2 - 2a - 1)Q(x + y) + 3[Q(x) + Q(y)] \end{aligned}$$

for all $x, y \in U$. We observe from these inequalities that $Q(\lambda x) = \lambda^2 Q(x)$ for $\lambda := a + 1, a - 1, 2, 2a - 1$, and for all $x \in U$. Replacing y by ay in (2.15) and switching x with y in the resulting equation, and then adding two equations side by side, we obtain by inductive assumption that

$$(2.16) \quad \begin{aligned} Q((a - 1)x + a^2y) + Q(a^2x + (a - 1)y) + (a^3 - 2a^2 + 2a + 2)Q(x - y) \\ = (a^3 - 2a - 2)Q(x + y) + (a^4 - a^2 + 2a + 5)[Q(x) + Q(y)] \end{aligned}$$

for all $x, y \in U$. Now we substitute $y := ay - x$ in (2.10) to get

$$(2.17) \quad \begin{aligned} & Q((a-1)x + ay) + Q((a-1)x - a^2y) + (a-1)Q(2x - ay) \\ &= a^2(a+1)Q(y) + (a^2-1)[Q(x) + Q(x-ay)] \end{aligned}$$

for all $x, y \in U$. Switching x with y in (2.17), and then adding two equations side by side, we obtain by virtue of (2.15), (2.16) and (2.10) that

$$(2.18) \quad \begin{aligned} & (a-1)[Q(2x - ay) + Q(ax - 2y)] + (3a^2 - 5a - 2)Q(x + y) \\ &= (a^3 + a^2 - 2a - 8)[Q(x) + Q(y)] + (a^2 + a + 2)Q(x - y) \end{aligned}$$

holds for all $x, y \in U$. Note from (2.18) that $Q(\lambda x) = \lambda^2 Q(x)$ for $\lambda := a-2, a+2$, and for all $x \in U$. Now substituting x for $2x$ in (2.10) yields

$$(2.19) \quad \begin{aligned} & Q(2ax + y) + Q(2x + ay) + (a-1)Q(2x - y) \\ &= (a+1)Q(2x + y) + (a^2 - 1)[4Q(x) + Q(y)] \end{aligned}$$

for all $x, y \in U$. Exchanging x and y in (2.19) and then adding two equations side by side, one obtains by (2.11), (2.18)

$$(2.20) \quad \begin{aligned} & (a-1)[Q(2ax + y) + Q(x + 2ay)] + (a^2 - a + 4)Q(x - y) \\ &= (4a^3 - 6a^2 + 3a + 7)[Q(x) + Q(y)] + (3a^2 - 3a - 4)Q(x + y) \end{aligned}$$

for all $x, y \in U$. We remark that $Q(\lambda x) = \lambda^2 Q(x)$ for $\lambda := 2a-1, 2a+1$, and for all $x \in U$

Now, let's transform by variables like as $x := 2ax + y$ and $y := x + 2ay$ in (2.15). Then we can rewrite the equation (2.15) in the form

$$(2.21) \quad \begin{aligned} & (2a-1)^2[Q((a+1)x + ay) + Q(ax + (a+1)y)] + (2a-1)^2Q(x - y) \\ &= (2a^2 - 2a - 1)(2a+1)^2Q(x + y) + 3[Q(2ax + y) + Q(x + 2ay)] \end{aligned}$$

for all $x, y \in U$. Multiplying both sides of (2.21) by $(a-1)$ and applying (2.20) to the resulting expression, we get

$$(2.22) \quad \begin{aligned} & (a-1)(2a-1)^2[Q((a+1)x + ay) + Q(ax + (a+1)y)] \\ & \quad + (a-1)(2a-1)^2Q(x - y) \\ &= (8a^5 - 8a^4 - 10a^3 + 13a^2 - 4a - 11)Q(x + y) \\ & \quad + 3(4a^3 - 6a^2 + 3a + 7)[Q(x) + Q(y)] - (4a^3 - 5a^2 + 2a + 11)Q(x - y) \end{aligned}$$

for all $x, y \in U$. Multiplying $(a-1)(2a-1)^2$ on both sides of (2.13) and applying (2.22) to the resulting expression, we get finally

$$\begin{aligned} & (a-1)(2a-1)^2[Q((a+1)x + y) + Q(x + (a+1)y)] \\ &= (4a^4 - 8a^2 + 6a + 10)Q(x + y) - (4a^4 - 8a^3 + 2a^2 + 2a - 12)Q(x - y) \\ & \quad + (4a^5 - 11a^3 + 3a^2 + 4a - 24)[Q(x) + Q(y)], \end{aligned}$$

which can be written in the form

$$\begin{aligned}
 (2.23) \quad & (a-1)(2a-1)^2 \left[Q((a+1)x+y) + Q(x+(a+1)y) + aQ(x-y) \right] \\
 & = (a-1)(2a-1)^2 \left[(a+2)Q(x+y) + ((a+1)^2-1)[Q(x)+Q(y)] \right] \\
 & \quad + (3a^2-3a+12)[Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y)]
 \end{aligned}$$

for all $x, y \in U$. Since Q satisfies the equation (2.10) for $a+1$, the last equation reduces to $Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y) = 0$. Consequently, we have proved that if Q satisfies the equation (2.10) for $a+1$, then Q satisfies the equation (1.1). Therefore, by induction argument the equation (2.10) implies the equation (1.1) for each positive integer $a > 1$.

Conversely, it is obvious that the equation (1.1) implies the functional equation (2.10). This completes the proof. \square

Theorem 2.4. *Let U and V be real vector spaces. A mapping $Q : U \rightarrow V$ satisfies the functional equation (1.5) for each positive integer $n > 2$ if and only if a mapping $Q : U \rightarrow V$ satisfies the functional equation (1.1). Thus there exists a symmetric biadditive mapping $B : U \times U \rightarrow V$ such that $Q(x) = B(x, x)$ for all $x \in U$.*

Proof. It is easy to see that the equation (1.1) implies the functional equation (1.5). Conversely, let Q satisfy the equation (1.5). Putting $x_1 := x$, $x_2 := y$ and $x_i := 0$ for all $i = 3, \dots, n$ in (1.5), we get

$$\begin{aligned}
 (2.24) \quad & Q(x-ay) + Q(ax-y) + (a-1)Q(x+y) \\
 & = (a+1)Q(x-y) + (a^2-1)[Q(x)+Q(y)]
 \end{aligned}$$

for all $x, y \in U$, where $a := n-1$ is a positive integer with $a \geq 2$. By the previous Lemma 2.3, the mapping $Q : U \rightarrow V$ satisfies the functional equation (1.1). \square

The following result is interesting and useful characterization formulas for an inner product space among normed linear spaces.

Corollary 2.5. *Let U be a normed linear space. Then the following statements are equivalent:*

- (a) U is an inner product space.
- (b) The norm in U satisfies the condition:

$$\begin{aligned}
 & \|ax+y\|^2 + \|x+ay\|^2 + (a-1)\|x-y\|^2 \\
 & = (a+1)\|x+y\|^2 + (a^2-1)(\|x\|^2 + \|y\|^2)
 \end{aligned}$$

for all $x, y \in U$ and for some fixed integer a with $|a| \neq 1$.

- (c)

$$n \sum_{1 \leq i < j \leq n} \|x_i - x_j\|^2 = \sum_{i=1}^n \left\| \sum_{j=1}^n x_j - nx_i \right\|^2$$

for all $x_i (i = 1, \dots, n) \in U$ and a fixed $n > 2$.

Proof. The proof is obvious by Lemma 2.3 and Theorem 2.4. The inner product is defined as usual by

$$\begin{aligned}(x, y) &= 1/4 (\|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2), \text{ and} \\ (x, y) &= 1/4 (\|x + y\|^2 - \|x - y\|^2)\end{aligned}$$

for the complex and real spaces, respectively. □

3. STABILITY OF FE. (1.5) IN QUASI-BANACH MODULES

Now let \mathcal{A} be a complex $*$ -algebra with unit and let M be a left \mathcal{A} -module. Let us call a mapping $Q : M \rightarrow \mathcal{A}$ an \mathcal{A} -quadratic mapping if both relations $Q(ax) = aQ(x)a^*$ and $Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y)$ are fulfilled [26]. A mapping $Q : M \rightarrow \mathcal{A}$ is called a *generalized \mathcal{A} -quadratic mapping* if $Q(ax) = aQ(x)a^*$ for all $x \in M$, and the following identity holds:

$$Q\left(\sum_{i=1}^n a_i x_i\right) + \sum_{1 \leq i < j \leq n} a_i a_j Q(x_i - x_j) = \left(\sum_{i=1}^n a_i\right) \left[\sum_{i=1}^n a_i Q(x_i)\right]$$

for all $x_i \in M$, some fixed a_i in \mathbb{R} ($i = 1, \dots, n$) and at least two of them are nonzero such that $\sum_{i=1}^n a_i \neq 0$, and a fixed $n \geq 2$ [15]. It was shown that the notion of \mathcal{A} -quadratic mapping is equivalent to the notion of generalized \mathcal{A} -quadratic mapping if all spaces are over the complex number field and a mapping $B : M \times M \rightarrow \mathcal{A}$ is defined in terms of the mapping Q as

$$(3.1) \quad B(x, y) = \frac{1}{4}[Q(x + y) - Q(x - y) + iQ(x + iy) - iQ(x - iy)]$$

for all x, y in M [15]. It was indicated in [26] that if the relation (3.1) holds and Q is an \mathcal{A} -quadratic form, then B is an \mathcal{A} -sesquilinear form and $Q(x) = B(x, x)$, and vice versa. Now it follows easily from Theorem 2.4 that a mapping Q is a generalized \mathcal{A} -quadratic mapping if and only if

$$\begin{aligned}Q(ax) &= aQ(x)a^*, \\ n \sum_{1 \leq i < j \leq n} Q(x_i - x_j) &= \sum_{i=1}^n Q\left(\sum_{j=1}^n x_j - nx_i\right)\end{aligned}$$

for all x and (x_1, \dots, x_n) , where $n \geq 3$.

Now we are ready to investigate the generalized Hyers-Ulam-Rassias stability problem for approximate \mathcal{A} -quadratic mappings acting on $\mathcal{U}(\mathcal{A})$ of the equation (1.5) in quasi-Banach modules over a C^* -algebra. Let M_1 and M_2 be quasi-Banach \mathcal{A} -bimodules and let $K \geq 1$ be the modulus of concavity of $\|\cdot\|$ throughout this section unless we give any specific reference. Given a mapping $f : M_1 \rightarrow M_2$, we

define a difference $D_u f : M_1^n \rightarrow M_2$ of the equation (1.5) as

$$D_u f(x_1, \dots, x_n) := n \sum_{1 \leq i < j \leq n} f(ux_i - ux_j) - \sum_{i=1}^n u f \left(\sum_{j=1}^n x_j - nx_i \right) u^*,$$

for all $x_i \in M_1$ and $u \in \mathcal{U}(\mathcal{A})$, which is called the approximate remainder of the functional equation (1.5) and acts as a perturbation of the equation.

Theorem 3.1. *Assume that there exists a mapping $\varphi : M_1^n \rightarrow [0, \infty) := \mathbb{R}_+$ for which a mapping $f : M_1 \rightarrow M_2$ satisfies the functional inequality*

$$(3.2) \quad \|D_u f(x_1, \dots, x_n)\| \leq \varphi(x_1, \dots, x_n)$$

for all $(x_1, \dots, x_n) \in M_1^n$ and for all $u \in \mathcal{U}(\mathcal{A})$, and the following series

$$(3.3) \quad \sum_{i=0}^{\infty} \frac{K^i \varphi((n-1)^i x_1, \dots, (n-1)^i x_n)}{(n-1)^{2i}} < \infty$$

for all $(x_1, \dots, x_n) \in M_1^n$. If either f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in M_1$, then there exists a unique generalized \mathcal{A} -quadratic mapping $Q : M_1 \rightarrow M_2$ which satisfies the equation (1.5) and the inequality

$$(3.4) \quad \left\| f(x) + \frac{(n-1)f(0)}{2} - Q(x) \right\| \leq \frac{K}{(n-1)^2} \sum_{i=0}^{\infty} \frac{K^i \Phi((n-1)^i x)}{(n-1)^{2i}}$$

for all $x \in M_1$, where

$$\begin{aligned} \Phi(x) &:= \min_{1 \leq i \leq n} \left\{ \varphi_i(-x) + \frac{|(n^2+1) - (i+1)n|}{n} \tilde{\varphi}(x) \right\}, \\ \varphi_i(x) &:= \varphi(0, \dots, 0, \underbrace{x}_{i\text{-th}}, 0, \dots, 0), \quad (i = 1, \dots, n), \end{aligned}$$

$$\text{and } \tilde{\varphi}(x) := \min_{1 \leq i \leq n-1} \{ \varphi_i(x) + \varphi_{i+1}(x) \}$$

for all $x \in M_1$. The mapping Q is defined by

$$Q(x) = \lim_{k \rightarrow \infty} \frac{f((n-1)^k x)}{(n-1)^{2k}}$$

for all $x \in M_1$.

Proof. Put $u := 1 \in \mathcal{U}(\mathcal{A})$ in (3.2). Then for each $i = 1, \dots, n-1$, interchanging x_i for x and x_j for 0 for all $j \neq i$ in (3.2) and then comparing the sequent inequalities, we get

$$n\|f(x) - f(-x)\| \leq \varphi_i(x) + \varphi_{i+1}(x)$$

for all $x \in M_1$ and for all $i = 1, \dots, n-1$. Thus one obtains the approximate even condition of f

$$(3.5) \quad \|f(x) - f(-x)\| \leq \frac{1}{n} \tilde{\varphi}(x), \quad \tilde{\varphi}(x) := \min_{1 \leq i \leq n-1} \{ \varphi_i(x) + \varphi_{i+1}(x) \}$$

for all $x \in M_1$. For each $i = 1, \dots, n$, replacing x_i by $-x$ and x_j by 0 for all $j \neq i$ we observe that

$$\begin{aligned} & \left\| (i-1)nf(x) + [n^2 - (i+1)n + 1]f(-x) + n \binom{n-1}{2} f(0) - f((n-1)x) \right\| \\ & \leq \varphi_i(-x) \end{aligned}$$

for all $x \in M_1$. Associating the last inequality with (3.5), we obtain

$$\begin{aligned} & \left\| (n-1)^2 f(x) + n \binom{n-1}{2} f(0) - f((n-1)x) \right\| \\ & \leq \varphi_i(-x) + \frac{|(n^2+1) - (i+1)n|}{n} \tilde{\varphi}(x) \end{aligned}$$

for all $x \in M_1$ and for all $i = 1, \dots, n$. Hence one has the following inequality

$$(3.6) \quad \left\| (n-1)^2 f(x) + n \binom{n-1}{2} f(0) - f((n-1)x) \right\| \leq \Phi(x)$$

for all $x \in M_1$. Define a mapping $g : M_1 \rightarrow M_2$ by $g(x) := f(x) + \frac{(n-1)f(0)}{2}$ for all $x \in M_1$. Then it follows from (3.6) that

$$(3.7) \quad \left\| g(x) - \frac{g((n-1)x)}{(n-1)^2} \right\| \leq \frac{1}{(n-1)^2} \Phi(x)$$

for all $x \in M_1$, from which we obtain by applying a standard procedure of the induction argument on m that

$$\begin{aligned} (3.8) \quad \left\| g(x) - \frac{g((n-1)^m x)}{(n-1)^{2m}} \right\| & \leq \frac{K}{(n-1)^2} \sum_{i=0}^{m-2} \left(\frac{K}{(n-1)^2} \right)^i \Phi((n-1)^i x) \\ & \quad + \frac{1}{(n-1)^2} \left(\frac{K}{(n-1)^2} \right)^{m-1} \Phi((n-1)^{m-1} x) \end{aligned}$$

for all $x \in M_1$ and all $m \geq 1$, which is considered to be (3.7) for $m = 1$. In fact, we figure out by the inequality (3.7),

$$\begin{aligned} & \left\| g(x) - \frac{g((n-1)^{m+1} x)}{(n-1)^{2(m+1)}} \right\| \\ & \leq K \left\| g(x) - \frac{g((n-1)x)}{(n-1)^2} \right\| + K \left\| \frac{g((n-1)x)}{(n-1)^2} - \frac{g((n-1)^{m+1} x)}{(n-1)^{2(m+1)}} \right\| \\ & \leq \frac{K}{(n-1)^2} \Phi(x) + \frac{K}{(n-1)^2} \left\| g((n-1)x) - \frac{g((n-1)^{m+1} x)}{(n-1)^{2m}} \right\|, \end{aligned}$$

which, in accordance with inductive assumption, yields (3.8) for $m + 1$. Thus one obtains that for all nonnegative integers m, l with $m > l$

$$\begin{aligned}
 (3.9) \quad & \left\| \frac{g((n-1)^l x)}{(n-1)^{2l}} - \frac{g((n-1)^m x)}{(n-1)^{2m}} \right\| \\
 &= \frac{1}{(n-1)^{2l}} \left\| g((n-1)^l x) - \frac{g((n-1)^{m-l} \cdot (n-1)^l x)}{(n-1)^{2(m-l)}} \right\| \\
 &\leq \frac{K}{(n-1)^{2l+2}} \sum_{i=0}^{m-l-2} \frac{K^i \Phi((n-1)^{l+i} x)}{(n-1)^{2i}} + \frac{1}{(n-1)^{2l+2}} \frac{K^{m-l-1} \Phi((n-1)^{m-1} x)}{(n-1)^{2(m-l-1)}} \\
 &\leq \frac{K}{K^l (n-1)^2} \sum_{i=l}^{m-2} \frac{K^i \Phi((n-1)^i x)}{(n-1)^{2i}} + \frac{1}{K^l (n-1)^2} \frac{K^{m-1} \Phi((n-1)^{m-1} x)}{(n-1)^{2(m-1)}},
 \end{aligned}$$

which tends to zero by (3.3) as $l \rightarrow \infty$. Hence the sequence $\left\{ \frac{g((n-1)^m x)}{(n-1)^{2m}} \right\}_{m \in \mathbb{N}}$ is a Cauchy sequence for any $x \in M_1$, and so it converges by the completeness of M_2 . Therefore we can define a mapping $Q : M_1 \rightarrow M_2$ by

$$Q(x) = \lim_{m \rightarrow \infty} \frac{g((n-1)^m x)}{(n-1)^{2m}} = \lim_{m \rightarrow \infty} \frac{f((n-1)^m x)}{(n-1)^{2m}}$$

for all $x \in M_1$. Taking the limit as $m \rightarrow \infty$ in (3.8), we obtain the desired inequality (3.4). Exchanging (x_1, \dots, x_n) for $((n-1)^m x_1, \dots, (n-1)^m x_n)$ in (3.2) and dividing both sides by $(n-1)^{2m}$, we have

$$\begin{aligned}
 (3.10) \quad & \|D_1 Q(x_1, \dots, x_n)\| \\
 &= \lim_{m \rightarrow \infty} \frac{1}{(n-1)^{2m}} \|Df((n-1)^m x_1, \dots, (n-1)^m x_n)\| \\
 &\leq \lim_{m \rightarrow \infty} \frac{K^m}{(n-1)^{2m}} \varphi((n-1)^m x_1, \dots, (n-1)^m x_n) \\
 &= 0.
 \end{aligned}$$

Therefore the mapping Q satisfies the equation (1.5) and hence Q is quadratic.

To prove the uniqueness, let Q' be another quadratic mapping satisfying (3.4). Then we get by Lemma 2.2 that $Q'((n-1)^m x) = (n-1)^{2m} Q'(x)$ for all $x \in M_1$ and all $m \in \mathbb{N}$. Thus we have

$$\begin{aligned}
 & \|Q(x) - Q'(x)\| \\
 &\leq \frac{1}{(n-1)^{2m}} \left\{ K \left\| Q((n-1)^m x) - f((n-1)^m x) - \frac{(n-1)f(0)}{2} \right\| \right. \\
 &\quad \left. + K \left\| f((n-1)^m x) + \frac{(n-1)f(0)}{2} - Q'((n-1)^m x) \right\| \right\} \\
 &\leq \frac{2K^2}{K^m (n-1)^2} \sum_{i=0}^{\infty} \frac{K^{m+i} \Phi((n-1)^{k+i} x)}{(n-1)^{2(m+i)}}
 \end{aligned}$$

for all $x \in M_1$. Taking the limit as $m \rightarrow \infty$, then we conclude that $Q(x) = Q'(x)$ for all $x \in M_1$.

Finally, we show that the quadratic mapping Q is \mathcal{A} -quadratic. Under the assumption that either f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in M_1$, the quadratic mapping Q satisfies $Q(tx) = t^2Q(x)$ for all $x \in M_1$ and all $t \in \mathbb{R}$ by the same reasoning as the proof of [6]. That is, Q is \mathbb{R} -quadratic. Putting $x_1 := -(n-1)^m x$ and $x_i := 0$ for all $i = 2, \dots, n$ in (3.2) and dividing the resulting inequality by $(n-1)^{2m}$,

$$\begin{aligned} & \frac{1}{(n-1)^{2m}} \left\| n(n-1)f(-(n-1)^m u x) + n \binom{n-1}{2} f(0) \right. \\ & \quad \left. - u f((n-1)^{m+1} x) u^* - (n-1) u f(-(n-1)^m x) u^* \right\| \\ & \leq \frac{K^m}{(n-1)^{2m}} \varphi(-(n-1)^m x, 0, \dots, 0). \end{aligned}$$

Taking the limit as $m \rightarrow \infty$ and using the evenness of Q , we see that $Q(ux) = uQ(x)u^*$ for all $x \in M_1$ and for each $u \in \mathcal{U}(\mathcal{A})$. The last relation is also true for $u = 0$. Now let a be a nonzero element in \mathcal{A} and L a positive integer greater than $4|a|$. Then we have $|\frac{a}{L}| < \frac{1}{4} < 1 - \frac{2}{3}$. By [14, Theorem 1], there exist three elements $u_1, u_2, u_3 \in \mathcal{U}(\mathcal{A})$ such that $3\frac{a}{L} = u_1 + u_2 + u_3$. Thus we calculate in conjunction with [13, Lemma 2.1] that

$$\begin{aligned} Q(ax) &= Q\left(\frac{L}{3} 3\frac{a}{L} x\right) = \left(\frac{L}{3}\right)^2 Q(u_1 x + u_2 x + u_3 x) \\ &= \left(\frac{L}{3}\right)^2 B(u_1 x + u_2 x + u_3 x, u_1 x + u_2 x + u_3 x) \\ &= \left(\frac{L}{3}\right)^2 (u_1 + u_2 + u_3) B(x, x) (u_1^* + u_2^* + u_3^*) \\ &= \left(\frac{L}{3}\right)^2 3\frac{a}{L} Q(x) 3\frac{a^*}{L} = aQ(x)a^* \end{aligned}$$

for all $a \in \mathcal{A} (a \neq 0)$ and for all $x \in M_1$. So the unique \mathbb{R} -quadratic mapping Q is also generalized \mathcal{A} -quadratic, as desired. This completes the proof. \square

Theorem 3.2. *Assume that the approximate remainder $D_u f$ of a mapping $f : M_1 \rightarrow M_2$ satisfies the functional inequality (3.2) for all $(x_1, \dots, x_n) \in M_1^n$ and all $u \in \mathcal{U}(\mathcal{A})$, and that the following series*

$$(3.11) \quad \sum_{i=1}^{\infty} K^i (n-1)^{2i} \varphi\left(\frac{x_1}{(n-1)^i}, \dots, \frac{x_n}{(n-1)^i}\right)$$

converges for all $(x_1, \dots, x_n) \in M_1^n$. If either f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in M_1$, then there exists a unique generalized \mathcal{A} -quadratic mapping $Q : M_1 \rightarrow M_2$ which satisfies the equation (1.5) and the inequality

$$(3.12) \quad \|f(x) - Q(x)\| \leq \frac{1}{(n-1)^2} \sum_{i=1}^{\infty} K^i (n-1)^{2i} \Phi\left(\frac{x}{(n-1)^i}\right)$$

for all $x \in M_1$, where Φ is defined as in Theorem 3.1. The mapping Q is defined by

$$Q(x) = \lim_{m \rightarrow \infty} (n-1)^{2m} f\left(\frac{x}{(n-1)^m}\right)$$

for all $x \in M_1$.

Proof. We use the same notations as those of Theorem 3.1. We observe that $\varphi(0, \dots, 0) = 0$ by the convergence (3.11), and thus we have $f(0) = 0$ by setting $x_i := 0$ in (3.2) for all $i = 1, \dots, n$. Now we get by (3.7)

$$\left\| f(x) - (n-1)^2 f\left(\frac{x}{n-1}\right) \right\| \leq \Phi\left(\frac{x}{n-1}\right), \quad x \in M_1,$$

which yields by induction

$$(3.13) \quad \left\| f(x) - (n-1)^{2m} f\left(\frac{x}{(n-1)^m}\right) \right\| \leq \frac{1}{(n-1)^2} \sum_{i=1}^{m-1} K^i (n-1)^{2i} \Phi\left(\frac{x}{(n-1)^i}\right) + \frac{K^m (n-1)^{2m}}{K(n-1)^2} \Phi\left(\frac{x}{(n-1)^m}\right)$$

for all $x \in M_1$ and all integers $m > 1$.

It follows by (3.11) that the sequence $\left\{ (n-1)^{2m} f\left(\frac{x}{(n-1)^m}\right) \right\}_{m \in \mathbb{N}}$ is a Cauchy sequence for any $x \in M_1$. Since M_2 is complete, we may define a mapping $Q : M_1 \rightarrow M_2$ by

$$Q(x) = \lim_{m \rightarrow \infty} (n-1)^{2m} f\left(\frac{x}{(n-1)^m}\right), \quad x \in M_1.$$

The rest of the proof goes through by the same way as that of Theorem 3.1. This completes the proof. \square

From the main Theorem 3.1 and Theorem 3.2 we obtain the following corollary concerning the stability of the equation (1.5).

Corollary 3.3. *Let r, ε be positive real numbers with $r-2 < -\log_{n-1} K$ or $r-2 > \log_{n-1} K$. Assume that a mapping $f : M_1 \rightarrow M_2$ satisfies the inequality*

$$(3.14) \quad \|D_u f(x_1, \dots, x_n)\| \leq \varepsilon \sum_{i=1}^n \|x_i\|^r$$

for all $x_i \in M_1$ and for all $u \in \mathcal{U}(\mathcal{A})$. Then there exists a unique generalized \mathcal{A} -quadratic mapping $Q : M_1 \rightarrow M_2$ which satisfies the equation (1.5) and the inequality

$$\|f(x) - Q(x)\| \leq \begin{cases} \frac{(n+2)K\varepsilon\|x\|^r}{n[(n-1)^2 - K(n-1)^r]}, & \text{if } r-2 < -\log_{n-1} K \\ \frac{(n+2)K\varepsilon\|x\|^r}{n[(n-1)^r - K(n-1)^2]}, & \text{if } r-2 > \log_{n-1} K \end{cases}$$

for all $x \in M_1$.

Proof. Define $\varphi(x_1, \dots, x_n) := \varepsilon(\|x_1\|^r + \dots + \|x_n\|^r)$ for all $(x_1, \dots, x_n) \in M_1^n$. Then we have the conditions $\tilde{\varphi}(x) := 2\varepsilon\|x\|^p$ and $\Phi(x) := \varepsilon\|x\|^p + \frac{2}{n}\varepsilon\|x\|^p$ for all $x \in M_1$. Applying Theorem 3.1 and Theorem 3.2, we obtain the desired results

according to the cases of r . Exchanging x_i for 0 in (3.14) for all $i = 1, \dots, n$ yields $f(0) = 0$. \square

Problem 3.4. *It is an open problem to investigate the stability problem of Ulam for the case of K and r with $-\log_{n-1} K \leq r - 2 \leq \log_{n-1} K$ in Corollary 3.3.*

Corollary 3.5. *Assume that there exists a nonnegative number θ for which a mapping $f : M_1 \rightarrow M_2$ satisfies the inequality*

$$\|D_u f(x_1, \dots, x_n)\| \leq \theta$$

for all $x_i \in M_1$ and for all $u \in \mathcal{U}(\mathcal{A})$. Then there exists a unique generalized \mathcal{A} -quadratic mapping $Q : M_1 \rightarrow M_2$ which satisfies the equation (1.5) and the inequality

$$\left\| f(x) + \frac{(n-1)f(0)}{2} - Q(x) \right\| \leq \frac{(n+2)K\theta}{n[(n-1)^2 - K]}, \quad \text{if } K < (n-1)^2$$

for all $x \in M_1$.

Problem 3.6. *If K is so large a constant that $(n-1)^2 \leq K$, then we can't guarantee that the functional equation (1.5) is stable on concerning the Ulam stability problem. So, it is interesting to investigate the stability problem of Ulam for the case of n, K with $(n-1)^2 \leq K$ in Corollary 3.5.*

4. STABILITY OF FE. (1.5) IN p -BANACH MODULES

We now prove the Hyers–Ulam–Rassias stability of the functional equation (1.5) in p -Banach \mathcal{A} -bimodules.

Theorem 4.1. *Let M_1 and M_2 be p -Banach \mathcal{A} -bimodules. Assume that the approximate remainder $D_u f$ of a mapping $f : M_1 \rightarrow M_2$ satisfies the functional inequality (3.2) for all $(x_1, \dots, x_n) \in M_1^n$ and all $u \in \mathcal{U}(\mathcal{A})$, and that the following series*

$$\sum_{i=0}^{\infty} \frac{\varphi((n-1)^i x_1, \dots, (n-1)^i x_n)^p}{(n-1)^{2ip}} < \infty$$

for all $(x_1, \dots, x_n) \in M_1^n$. If either f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in M_1$, then there exists a unique generalized \mathcal{A} -quadratic mapping $Q : M_1 \rightarrow M_2$ which satisfies the equation (1.5) and the inequality

$$\left\| f(x) + \frac{(n-1)f(0)}{2} - Q(x) \right\| \leq \frac{1}{(n-1)^2} \left[\sum_{i=0}^{\infty} \frac{\Phi((n-1)^i x)^p}{(n-1)^{2ip}} \right]^{1/p}$$

for all $x \in M_1$, where Q and Φ are defined as in Theorem 3.1.

Proof. It follows by the inequality (3.7) and the definition of p -norm that

$$\left\| \frac{g((n-1)^i x)}{(n-1)^{2i}} - \frac{g((n-1)^{i+1} x)}{(n-1)^{2(i+1)}} \right\|^p \leq \frac{1}{(n-1)^{2p}} \frac{1}{(n-1)^{2pi}} \Phi((n-1)^i x)^p,$$

and so

$$\begin{aligned} \left\| \frac{g((n-1)^l x)}{(n-1)^{2l}} - \frac{g((n-1)^m x)}{(n-1)^{2m}} \right\|^p &\leq \sum_{i=l}^{m-1} \left\| \frac{g((n-1)^i x)}{(n-1)^{2i}} - \frac{g((n-1)^{i+1} x)}{(n-1)^{2(i+1)}} \right\|^p \\ &\leq \frac{1}{(n-1)^{2p}} \sum_{i=l}^{m-1} \frac{1}{(n-1)^{2pi}} \Phi((n-1)^i x)^p \end{aligned}$$

for all $x \in M_1$ and all integers l, m with $m > l \geq 0$. Note that the series $\sum_{i=0}^{\infty} \frac{\Phi((n-1)^i x)^p}{(n-1)^{2ip}}$ converges for all $x \in M_1$. Thus we obtain the desired results using the similar argument to Theorem 3.1. \square

Theorem 4.2. *Let M_1 and M_2 be p -Banach \mathcal{A} -bimodules. Assume that the approximate remainder $D_u f$ of a mapping $f : M_1 \rightarrow M_2$ satisfies the functional inequality (3.2) for all $(x_1, \dots, x_n) \in M_1^n$ and all $u \in \mathcal{U}(\mathcal{A})$, and that the following series*

$$\sum_{i=1}^{\infty} (n-1)^{2ip} \varphi \left(\frac{x_1}{(n-1)^i}, \dots, \frac{x_n}{(n-1)^i} \right)^p < \infty$$

for all $(x_1, \dots, x_n) \in M_1^n$. If either f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in M_1$, then there exists a unique generalized \mathcal{A} -quadratic mapping $Q : M_1 \rightarrow M_2$ which satisfies the equation (1.5) and the inequality

$$\|f(x) - Q(x)\| \leq \frac{1}{(n-1)^2} \left[\sum_{i=1}^{\infty} (n-1)^{2ip} \Phi \left(\frac{x}{(n-1)^i} \right)^p \right]^{1/p}$$

for all $x \in M_1$, where Q and Φ are defined as in Theorem 3.2.

Corollary 4.3. *Let M_1 and M_2 be p -Banach \mathcal{A} -bimodules. Let r, ε be positive real numbers with $r \neq 2$. Assume that a mapping $f : M_1 \rightarrow M_2$ satisfies the inequality*

$$\|D_u f(x_1, \dots, x_n)\| \leq \varepsilon \sum_{i=1}^n \|x_i\|^r$$

for all $x_i \in M_1$ and for all $u \in \mathcal{U}(\mathcal{A})$. Then there exists a unique generalized \mathcal{A} -quadratic mapping $Q : M_1 \rightarrow M_2$ which satisfies the equation (1.5) and the inequality

$$\|f(x) - Q(x)\| \leq \begin{cases} \frac{(n+2)\varepsilon\|x\|^r}{n[(n-1)^{2p} - (n-1)^{rp}]^{1/p}}, & \text{if } r < 2 \\ \frac{(n+2)\varepsilon\|x\|^r}{n[(n-1)^{rp} - (n-1)^{2p}]^{1/p}}, & \text{if } r > 2 \end{cases}$$

for all $x \in M_1$.

Remark 4.4. The result for the case $K = 1$ in Theorem 3.1 (Theorem 3.2, Corollary 3.3, respectively) is the same as the result for the case $p = 1$ in Theorem 4.1 (Theorem 4.2, Corollary 4.3, respectively).

Let M_1 and M_2 be Banach left \mathcal{A} -modules and let $\hat{a} := aa^*, a^*a$, or $\frac{aa^* + a^*a}{2}$ for each $a \in \mathcal{A}$. A mapping $Q : M_1 \rightarrow M_2$ is called \mathcal{A}_{sa} -quadratic if $Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y)$ and $Q(ax) = \hat{a}Q(x)$ for all $a \in \mathcal{A}$ and all $x, y \in M_1$ [16]. Since two Banach spaces E_1 and E_2 are considered as Banach modules over $\mathcal{A} := \mathbb{C}$, the \mathcal{A}_{sa} -quadratic mapping $Q : E_1 \rightarrow E_2$ implies $Q(ax) = |a|^2 Q(x)$ for all $a \in \mathbb{C}$.

Theorem 4.5. *Let M_1 and M_2 be quasi-Banach \mathcal{A} -bimodules. Assume that there exists a mapping $\varphi : M_1^n \rightarrow \mathbb{R}_+$ for which a mapping $f : M_1 \rightarrow M_2$ satisfies the functional inequality*

$$\left\| n \sum_{1 \leq i < j \leq n} f(ux_i - ux_j) - \sum_{i=1}^n \hat{u} f \left(\sum_{j=1}^n x_j - nx_i \right) \right\| \leq \varphi(x_1, \dots, x_n), \quad \forall x_i \in M_1, \forall u \in \mathcal{A}(|u| = 1),$$

and the series (3.3) converges for all $x_i \in M_1, i = 1, \dots, n$. If either f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in M_1$, then there exists a unique \mathcal{A}_{sa} -quadratic mapping $Q : M_1 \rightarrow M_2$, defined by $Q(x) = \lim_{m \rightarrow \infty} \frac{f((n-1)^m x)}{(n-1)^{2m}}$, which satisfies the equation (1.5) and the inequality (3.4) for all $x \in M_1$.

Proof. By the same reasoning as the proof of Theorem 3.1, it follows from $u = 1 \in \mathcal{A}(|u| = 1)$ that there exists a unique \mathbb{R} -quadratic mapping $Q : M_1 \rightarrow M_2$, defined by $Q(x) = \lim_{m \rightarrow \infty} \frac{f((n-1)^m x)}{(n-1)^{2m}}$, which satisfies the equation (1.5) and the inequality (3.4). By the similar manner to the proof of Theorem 3.1 we obtain that $Q(ux) = \hat{u}Q(x)$ for all $x \in M_1$ and each $u \in \mathcal{A}(|u| = 1)$. The last relation is also true for $u = 0$. Since Q is \mathbb{R} -quadratic, for each element $a(a \neq 0) \in \mathcal{A}$

$$\begin{aligned} Q(ax) &= Q\left(|a| \frac{a}{|a|} x\right) = |a|^2 Q\left(\frac{a}{|a|} x\right) = |a|^2 \frac{\hat{a}}{|a|^2} Q(x) \\ &= \hat{a} Q(x) \end{aligned}$$

for all $x \in M_1$ and for all $a \in \mathcal{A}$. So the unique \mathbb{R} -quadratic mapping Q is also \mathcal{A}_{sa} -quadratic, as desired. This completes the proof. \square

Theorem 4.6. *Let M_1 and M_2 be quasi-Banach \mathcal{A} -bimodules. Assume that there exists a mapping $\varphi : M_1^n \rightarrow \mathbb{R}_+$ for which a mapping $f : M_1 \rightarrow M_2$ satisfies the functional inequality*

$$\left\| n \sum_{1 \leq i < j \leq n} f(ux_i - ux_j) - \sum_{i=1}^n \hat{u} f \left(\sum_{j=1}^n x_j - nx_i \right) \right\| \leq \varphi(x_1, \dots, x_n), \quad \forall x_i \in M_1, \forall u \in \mathcal{A}(|u| = 1),$$

and the series (3.11) converges for all $x_i \in M_1, i = 1, \dots, n$. If either f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in M_1$, then there exists a unique \mathcal{A}_{sa} -quadratic mapping $Q : M_1 \rightarrow M_2$, defined by $Q(x) = \lim_{m \rightarrow \infty} (n-1)^{2m} f\left(\frac{x}{(n-1)^m}\right)$, which satisfies the equation (1.5) and the inequality (3.12) for all $x \in M_1$.

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